MS221

Assignment Booklet II

Contents | Cut-off date

Please send all your answers to each tutor-marked assignment (TMA), together with a completed assignment form (PT3), to reach your tutor on or before the appropriate cut-off date shown above.

Be sure to fill in the correct assignment number on the PT3 form, and allow sufficient time in the post for each assignment to reach its destination on or before the cut-off date.

You are advised to keep a copy of your assignment in case of loss in the mail. Also keep all your marked assignments as you may need to make reference to them in later assignments or when you revise for the examination.

SUP 85564 5

Points to note when preparing solutions to TMA questions

- Contact your tutor if the meaning of any part of a question does not seem clear.
- Your solutions should not involve the use of Mathcad, except in those parts of questions where this is explicitly required or suggested.
- Clarity and accuracy of presentation are important in these assignments, so make sure that you express your answers as precisely as you can, giving detailed explanations where appropriate.
- Where a question involves mathematical calculation, show all your working. You may not receive full marks for a correct final answer that is not supported by working. You may receive some marks for working even if your final answer is incorrect or your solution is incomplete.
- Whenever you perform a calculation using a numerical answer found earlier, you should use the full-calculator-accuracy version of the earlier answer to avoid rounding errors.
- Number all of your pages, including any computer printouts.
- Indicate in each solution the page numbers of any computer printouts associated with that solution.
- The marks allocated to the parts of the questions are indicated in brackets in the margin. Each TMA is marked out of 100. Your overall score for a TMA will be the sum of your marks for the questions.

TMA MS221 03 **Cut-off date** 2 August 2006

This assignment covers *Block C*.

Question 1 – 15 marks

You should be able to answer this question after working through Chapter C1.

(a) Differentiate each of the following functions, identifying any general rules of calculus that you use.

(i) $f(x) = \tan(3x)\ln(\sin(3x))$ [*4*]

(ii) $f(x) = \dfrac{e^{-\sqrt{x}}}{x^3 + 1}$ [*4*]

(b) Consider the function $f(x) = x - 4\ln x$.

(i) Show that the equation $f(x) = 0$ has a solution in the interval $(1, 2)$. [*1*]

(ii) Show that for this function f the Newton–Raphson formula can be expressed as

$$x_{n+1} = \frac{4x_n(1 - \ln x_n)}{4 - x_n} \qquad (n = 0, 1, 2, \ldots).$$ [*4*]

(iii) Use Mathcad file 221C1-03 to find, to 9 decimal places, an approximate solution of the equation $f(x) = 0$ in the interval $(1, 2)$.

Provide a printout of your work. [*2*]

Question 2 – 20 marks

You should be able to answer this question after working through Chapter C1.

Use the graph-sketching strategy in Section 3 to sketch the graph of the function

$$f(x) = \frac{4x - 15}{x^2 - 9}.$$ [*20*]

Question 3 – 10 marks

You should be able to answer this question after working through Chapter C2.

Find each of the following indefinite integrals.

(a) $\displaystyle\int x^{5/2}\ln x\,dx \qquad (x > 0)$ [*5*]

(b) $\displaystyle\int \frac{(\arcsin x)^2}{\sqrt{1 - x^2}}\,dx \qquad (-1 < x < 1)$ [*5*]

Question 4 – 25 marks

You should be able to answer this question after working through Chapter C2.

(a) (i) Explain why the graph of the function $f(x) = (x - 3)\left(e^{-x/4} - 1\right)$ lies above the x-axis for $0 < x < 3$ and below the x-axis for $x > 3$. [*2*]

(ii) Use this fact to find the area enclosed by the graph and the x-axis between $x = 0$ and $x = 6$, giving your answer to 4 decimal places. [*12*]

(b) Find the volume of the solid of revolution obtained when the region under the graph of $f(x) = \sqrt{\sin x}/(\cos x)$, from $x = \frac{1}{6}\pi$ to $x = \frac{1}{3}\pi$, is rotated about the x-axis. Give your answer to 4 decimal places. [*7*]

(c) Starting with a blank worksheet, use Mathcad to find the indefinite integral in Question 3(a) above, and also to evaluate the volume described in part (b) of the current question to 4 decimal places.

Provide a printout of your work. [*4*]

Question 5 – 15 marks

You should be able to answer this question after working through Chapter C3.

In this question, f and g are the functions given by

$$f(x) = \frac{36}{(6+x)^2} \quad \text{and} \quad g(x) = \frac{36}{(1+x)^2}.$$

(a) By writing

$$\frac{36}{(6+x)^2} = \frac{1}{(1+\frac{1}{6}x)^2},$$

and using substitution in one of the standard Taylor series given in the course, find the Taylor series about 0 for f. Give explicitly all terms up to the term in x^3. Determine a range of validity for this Taylor series. [5]

(b) Use your answer to part (a), and the fact that $1 + x = 6 + (x - 5)$, to find the Taylor series about 5 for g. Give explicitly the same number of terms as in part (a). Determine a range of validity for this Taylor series. [4]

(c) Check the first four terms in the Taylor series that you found in part (b) by finding the first, second and third derivatives of g, and using these to find the cubic Taylor polynomial about 5 for g. [6]

Question 6 – 15 marks

You should be able to answer this question after working through Chapter C3.

(a) Use Taylor polynomials about 0 to evaluate $\cos(0.52)$ to 4 decimal places, showing all your working. [5]

(b) (i) Use multiplication of Taylor series to find the quartic Taylor polynomial about 0 for the function

$$f(x) = (e^x - x)\ln(1+x),$$

evaluating the coefficients. [4]

For parts (b)(ii) and (b)(iii), you should provide a printout showing your answers.

(ii) Verify your answer to part (b)(i) by using Mathcad to find the same Taylor polynomial. [1]

(iii) Use Mathcad to produce graphs of the function f and its quartic Taylor polynomial about 0 on the same axes. The scales on the x- and y-axes should run from -1 to 2 and from -3 to 3, respectively. [4]

(iv) Estimate from your graph the interval over which the quartic polynomial about 0 for f appears to approximate f closely. [1]

This assignment covers *Block D.*

Question 1 – 10 marks

You should be able to answer this question after working through Chapter D1.

(a) Let $w = 7 + i$ and $z = 2 - 3i$. Find each of the following, expressing complex numbers in Cartesian form.

(i) $|z|$ (ii) $\overline{z}$ (iii) z^{-1} (iv) wz (v) $\dfrac{w}{z}$ [8]

(b) Find a polynomial in z with real coefficients whose roots are $-5 - 2i$ and $-5 + 2i$. [2]

Question 2 – 15 marks

You should be able to answer this question after working through Chapter D1.

(a) Express the complex number -27 in polar form. [1]

(b) Calculate all the sixth roots of -27 and express them in polar form, using the principal value of the argument in each case. Show that these sixth roots form three pairs of conjugate complex numbers.

(*You may find it helpful to label the roots* $z_0, z_1, \ldots, z_5$.) [10]

(c) Hence factorise the polynomial $x^6 + 27$ into polynomial factors with real coefficients, where the factors are either linear or quadratic. You should replace any trigonometric expressions by their exact values, using surds. [4]

Question 3 – 5 marks

You should be able to answer this question after working through Chapter D2.

Use the rules in *Chapter D2*, Section 2, to find the remainders when the number

5312 4929 3007 1602

is divided by:

(a) 9; (b) 11; (c) 18; (d) 22.

Remember to show all of your working. [5]

0

Question 4 – 25 marks

You should be able to answer this question after working through Chapter D2.

(a) Give an example of a number x in $\mathbb{Z}_{30}$, other than 1 or 17, that has a multiplicative inverse in $\mathbb{Z}_{30}$. Without listing them all, give a description of the set of numbers in $\mathbb{Z}_{30}$ that have a multiplicative inverse in $\mathbb{Z}_{30}$. [2]

(b) Use Euclid's Algorithm to find the multiplicative inverse of 17 in $\mathbb{Z}_{30}$. [6]

A message in English is coded numerically using the correspondence below.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

It is then enciphered using the rule

$$E_{17}(m) = m^{17} \pmod{31}.$$

The resulting ciphertext is

$$\langle 18, 14, 25, 25, 10\rangle.$$

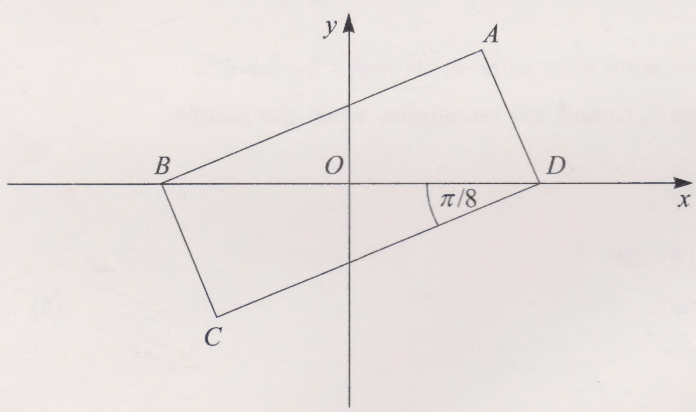
(c) Verify that the rule given satisfies the conditions for an exponential cipher. [2]

(d) Using the repeated squaring technique described in *Chapter D2*, decipher the ciphertext and find the message. [15]

Question 5 – 20 marks

You should be able to answer this question after working through Chapter D3.

In the figure below, $ABCD$ is a rectangle (Q) centred at the origin. The acute angle between the edge CD and the x-axis is $\pi/8$, as shown in the figure.



(a) (i) Using standard notation, write down the elements of the symmetry group $S(Q)$ of Q, giving a brief description of the geometric effect of each symmetry on points in the plane. [5]

(ii) Compile a Cayley table for $S(Q)$ under composition of symmetries. [4]

(b) (i) Show that the set $G = \{1, 8, 17, 19\}$ is a group under the operation $\times_{45}$. You should state the inverse of each element in $(G, \times_{45})$. [9]

(ii) Decide whether $(S(Q), \circ)$ is isomorphic to $(G, \times_{45})$, justifying your answer briefly. [2]

Question 6 – 15 marks

You should be able to answer this question after working through Chapter D4.

(a) One of the following propositions is true and one is false.

(A) The number $n^2 + 6n + 5$ is divisible by 3, for all $n \in \mathbb{N}$.

(B) The equation $x^3 = 2$ has no solution in $\mathbb{Z}_7$.

(i) State which of the two propositions is false, and give a counter-example to prove that it is false. [2]

(ii) Give a proof that the other proposition is true. [3]

(b) The variable propositions $a(n)$, $b(n)$, $c(n)$ and $d(n)$, where $n \in \mathbb{N}$, have the meanings given below.

$a(n)$ means: n is divisible by 5.
$b(n)$ means: n is divisible by 14.
$c(n)$ means: n is divisible by 20.
$d(n)$ means: n is divisible by 105.

(i) For each of $a(n)$, $b(n)$, $c(n)$ and $d(n)$, say whether it is necessary but not sufficient, sufficient but not necessary, necessary and sufficient, or neither necessary nor sufficient, in order that n be divisible by 35. [4]

(ii) Using a combination of just two of the propositions above, give a condition that is necessary and sufficient for n to be divisible by 140. [2]

(iii) Express in English the proposition

(C) $(a(n) \wedge b(n)) \Rightarrow d(n)$.

What is the converse of proposition (C)? State this converse both in symbols and in English. Give a number $n \in \mathbb{N}$ that is a counter-example showing that proposition (C) is false. [4]

Question 7 – 10 marks

You should be able to answer this question after working through Chapter D4.

(a) Use mathematical induction to prove that

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)} \qquad \text{for } n = 1, 2, 3, \ldots.$$ [8]

(b) Does the sum have a limit for large values of n? Justify your answer briefly. [2]

Printed in the United Kingdom by
The Open University